Recall from previous math courses that the reciprocal of any number \( x \) is \( \frac{1}{x} \).

For example, the reciprocal of 5 is \( \frac{1}{5} \) and the reciprocal of 0.5 is \( \frac{1}{0.5} \), or 2.

Throughout this course you have studied many connections between polynomial functions and real numbers. Does it follow then that polynomial functions also have reciprocals? Is the reciprocal also a polynomial? Is it a function? How would the graph and table of values of \( \frac{1}{f(x)} \) compare to the original function \( f(x) \)?

To begin answering these questions, consider the reciprocal of the basic linear function \( f(x) = x \). The reciprocal can be defined as \( g(x) = \frac{1}{f(x)} \), or simply \( g(x) = \frac{1}{x} \).

1. Consider the graph and table of values for \( f(x) = x \). The domain of \( f(x) \) is \((-\infty, \infty)\).

The points \((-1, -1)\) and \((1, 1)\) are shown and used to create three intervals for analysis.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = x )</td>
<td>-6</td>
<td>-5</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccc}
-1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & -\frac{1}{6}
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

\[
\begin{array}{cccccc}
g(x) = \frac{1}{x}
\end{array}
\]

\[
\begin{array}{cccccc}
\frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1
\end{array}
\]

\[
\begin{array}{cccccc}
f(x) = x
\end{array}
\]

\[
\begin{array}{cccccc}
-1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & 0 & \frac{1}{100} & \frac{1}{10} & \frac{1}{2} & 1
\end{array}
\]

\[
\begin{array}{cccccc}
g(x) = \frac{1}{x}
\end{array}
\]

\[
\begin{array}{cccccc}
-1 & -2 & -10 & -100 & \text{und} & 100 & 10 & 2 & 1
\end{array}
\]

a. Complete the table of values for \( g(x) = \frac{1}{x} \). Then plot the points and draw a smooth curve to graph \( g(x) \) on the coordinate plane.
b. Describe the graph of \( g(x) \). How is it similar to the graphs of other functions that you've studied? How is it different?

- One graph is increasing if the other is decreasing.
- The graph is curved.
- The graphs mirror each other.

Describe the end behavior of \( g(x) \). Explain your reasoning in terms of the graph, equation, and table of values.

- Does not touch the \( x \)- or \( y \)-axis.

The point at \( g(0) \) is said to be undefined because it is impossible to divide by 0.

d. Describe \( g(x) \) as \( x \) approaches 0 from the left. Explain the output behavior of the function in terms of the equation.

The function is decreasing. When I divide by a negative number that approaches 0, this results in output (\( y \)) values that approach negative infinity.

e. Describe \( g(x) \) as \( x \) approaches 0 from the right. Explain the output behavior of the function in terms of the equation.

The function is increasing. When I divide by small positive numbers that approach 0, this results in very large output (\( y \)) values that approach infinity.
c. Describe the domain and range of \( g(x) \).

\[ \text{D: all real } \#s \text{ except } 0 \quad \{ \mathbb{R}, x \neq 0 \} \]

\[ \text{R: all real } \#s \text{ except } 0 \quad \{ \mathbb{R}, y \neq 0 \} \]

Previously, you have represented the domain and range of functions using words, inequalities, or interval notation. You can also use set notation to represent these characteristics of functions.

Suppose that both the domain and range of a rational function, \( g(x) \), are represented verbally as "all real numbers except for 4." You can use inequalities, interval notation, and set notation to represent this verbal statement.

<table>
<thead>
<tr>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inequalities:</strong></td>
<td>( x &gt; 4 ) or ( x &lt; 4 )</td>
</tr>
<tr>
<td><strong>Interval notation:</strong></td>
<td>( (-\infty, 4) \cup (4, \infty) )</td>
</tr>
<tr>
<td><strong>Set notation:</strong></td>
<td>( { x \mid x \neq 4 } )</td>
</tr>
<tr>
<td><strong>( g(x) &gt; 4 ) or ( g(x) &lt; 4 )</strong></td>
<td>( (-\infty, 4) \cup (4, \infty) )</td>
</tr>
<tr>
<td><strong>( g(x) \mid g(x) \neq 4 )</strong></td>
<td>( { g(x) \mid g(x) \neq 4 } )</td>
</tr>
</tbody>
</table>

### "all real \#s except for 0"

d. Write your answer to Question 3, part (c), using each of the notations shown.

<table>
<thead>
<tr>
<th>Inequalities</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &gt; 0 )</td>
<td>( x &lt; 0 )</td>
<td>( g(x) &gt; 0 ) or ( g(x) &lt; 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interval Notation</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, 0) \cup (0, \infty) )</td>
<td>( (-\infty, 0) \cup (0, \infty) )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set Notation</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { x \mid x \neq 0 } )</td>
<td>( { g(x) \mid g(x) \neq 0 } )</td>
<td></td>
</tr>
</tbody>
</table>
The function \( g(x) = \frac{1}{x} \) is an example of a rational function. A rational function is any function that can be written as the ratio of two polynomials. It can be written in the form \( f(x) = \frac{P(x)}{Q(x)} \) where \( P(x) \) and \( Q(x) \) are polynomial functions, and \( Q(x) \neq 0 \). You have already seen some specific types of rational functions. Linear, quadratic, cubic, and higher order polynomial functions are types of rational functions.

Recall from your study of exponential functions that a horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects. In this problem, the function \( g(x) \) has a horizontal asymptote at \( y = 0 \).

Given the horizontal asymptote, the range of the rational function \( g(x) = \frac{1}{x} \) can be expressed:

- verbally: all real numbers except for 0
- as a compound inequality: \( g(x) > 0 \) or \( g(x) < 0 \).
- in interval notation: \((-\infty, 0) \cup (0, \infty)\).
- in set notation: \( \{ g(x) \mid g(x) \neq 0 \} \).

The function \( g(x) = \frac{1}{x} \) has a vertical asymptote at \( x = 0 \). A vertical asymptote is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0.

The vertical asymptote is often represented in textbooks and graphing calculators as a dashed or solid line. The convention used in this textbook is to represent asymptotes as dashed lines.

Given the vertical asymptote, the domain of the rational function \( g(x) = \frac{1}{x} \) can be expressed:

- verbally: all real numbers except for 0
- as a compound inequality: \( x > 0 \) or \( x < 0 \).
- in interval notation: \((-\infty, 0) \cup (0, \infty)\).
- in set notation: \( \{ x \mid x \neq 0 \} \).

Changing the mode to “dot” on many calculators removes the asymptote from the screen. Asymptotes are often more easily viewed with a smaller viewing window. Try \([-5, 5] \times [-5, 5]\) for \( g(x) \).
4. Analyze each function.

- \( f(x) = x \)
- \( g(x) = \frac{3x}{2} \)
- \( h(x) = \frac{\sqrt{x}}{2x} \)
- \( p(x) = \frac{3}{x} + 2 \)
- \( k(x) = 12 \)
- \( n(x) = \frac{2^x}{5} \)
- \( j(x) = \frac{4x^2 + 3x + 2}{6x^3 + 10} \)
- \( m(x) = \frac{1}{(x + 2)(x - 3)} \)

a. Circle the rational functions.

b. Explain why the remaining functions are not rational.

- \( h(x) \): The square root of \( x \) is not a polynomial
- \( n(x) \): has a variable as an exponent

c. Do you think the graphs of all rational functions will have a vertical asymptote? Explain your reasoning.

- No, if a rational function does not have a variable in the denominator, it will not have a V.A.

You have already explored transformations with functions in many function families. You learned that transformations performed on any function \( f(x) \) to form a new function \( g(x) \) can be described by the transformational function:

\[ g(x) = Af(B(x - C)) + D \]

Recall that this transformational function generalizes to any function. Changes to the \( A- \) or \( D \)-values dilate, translate, or reflect a function vertically. Changes to the \( B- \) or \( C \)-values dilate, translate, or reflect a function horizontally.

For a rational function, the transformational function can be written as:

\[ r(x) = A\left(\frac{1}{B(x - C)}\right) + D \]

5. Determine how the values of \( A, B, C, \) or \( D \) transform the graph of \( f(x) = \frac{1}{x} \).

a. \( g(x) = A\left(\frac{1}{x}\right) \), for positive and negative values of \( A \).

- \( A > 1 \): Vertical stretch
- \( 0 < A < 1 \): Vertical compression
- \( -(A) \): Reflection
b. \( h(x) = \frac{1}{x} + D \), for positive and negative values of \( D \).

\[ +D: \text{ Up} \]
\[ -D: \text{ Down} \]

c. \( j(x) = \frac{-1}{x-C} \), for positive and negative values of \( C \).

\[ x-C: \text{ Right} \]
\[ x+C: \text{ Left} \]

\[ k(x) = f(x) = \frac{1}{Bx} \], for positive and negative values of \( B \).

6. Use a graphing calculator to explore various rational functions of the form \( p(x) = \frac{a}{x} \) where \( a \) is a constant.

a. Describe changes in the function for various \( a \)-values. Make as many conjectures as you can about the key characteristics of functions in this form.

<table>
<thead>
<tr>
<th>( a )-value</th>
<th>new equation</th>
<th>changes?</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>( p(x) = \frac{23}{x} )</td>
<td>stretch</td>
</tr>
<tr>
<td>7</td>
<td>( p(x) = \frac{7}{x} )</td>
<td>stretch</td>
</tr>
<tr>
<td>-3</td>
<td>( p(x) = \frac{-3}{x} )</td>
<td>reflection, stretch</td>
</tr>
</tbody>
</table>
Recall that power functions are any functions of the form $y = x^n$ for $n \geq 1$. In Problem 1, My World Is Turned Upside Down, you discovered that the graph of the function $g(x) = \frac{1}{x}$ looks very different than the linear function $f(x) = x$. How will the graphs of the other power functions compare to their reciprocals? Will they all have the same shape? Will they all have asymptotes?

1. Analyze the graph of the quadratic power function $q(x) = x^2$.

Predict the graph of $r(x) = \frac{1}{x^2}$. Sketch it on the coordinate plane. Explain your reasoning.
2. Consider the graph and table of values for \( q(x) = x^2 \). The domain of \( q(x) \) is \((-\infty, \infty)\). The tables represent three intervals of the domain.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>( x )</th>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q(x) = x^2 )</td>
<td>25</td>
<td>16</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>( r(x) = \frac{1}{x^2} )</td>
<td>( r(x) = \frac{1}{x^2} )</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{9} )</td>
<td>( \frac{1}{16} )</td>
<td>( \frac{1}{25} )</td>
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<td>( \frac{1}{9} )</td>
<td>( \frac{1}{4} )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Complete the table of values for \( r(x) = \frac{1}{x^2} \).

b. Plot the points and sketch the reciprocal function \( r(x) \) on the coordinate plane.