Radicals can produce imaginary results. For example, the square root of $-4$ is equal to $2i$, $\sqrt{-4} = 2i$. But, in this chapter we are not going to talk about imaginary numbers. We are going to keep it real!
Previously, you have rewritten radicals by extracting roots involving numbers. In this lesson, you will explore how to extract roots for expressions of the form \( \sqrt[n]{x^n} \). To determine how to extract a variable from a radical, let’s consider several different values of \( n \).

1. For each value of \( n \) for the expression \( \sqrt[n]{x^n} \), complete the table and sketch the graph. Then identify the function family associated with the graph and write the corresponding equation.

   a. Let \( n = 2 \).

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   x & x^n = x^2 & \sqrt[n]{x^n} = \sqrt{2}x^2 \\
   \hline
   -2 & & \\
   -1 & & \\
   0 & & \\
   1 & & \\
   2 & & \\
   \hline
   \end{array}
   \]

   Function family of the graph: ________________

   Equation of the graph: ________________

   b. Let \( n = 3 \).

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   x & x^n = x^3 & \sqrt[n]{x^n} = \sqrt[3]{3}x^3 \\
   \hline
   -2 & & \\
   -1 & & \\
   0 & & \\
   1 & & \\
   2 & & \\
   \hline
   \end{array}
   \]

   Function family of the graph: ________________

   Equation of the graph: ________________
c. Let $n = 4$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^n = x^4$</th>
<th>$\sqrt[n]{x^n} = \sqrt[4]{x^4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Function family of the graph: _________________
Equation of the graph: _________________

d. Let $n = 5$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^n = x^5$</th>
<th>$\sqrt[n]{x^n} = \sqrt[5]{x^5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Function family of the graph: _________________
Equation of the graph: _________________
e. Analyze your representations for each value of \( n \). What do you notice?

To extract a variable from a radical, the expression \( \sqrt[n]{x^7} \) can be written as:

\[
\sqrt[n]{x^7} = \begin{cases} 
|x|, & \text{when } n \text{ is even} \\
|\sqrt[n]{x^7}|, & \text{when } n \text{ is odd}
\end{cases}
\]

2. Explain why \( \sqrt[n]{x^7} = |x| \) is incorrect, for real values of \( x \).

3. Asia and Melissa shared their work for extracting the root from \( \sqrt[7]{x^4} \), for real values of \( x \).

\[
\begin{align*}
\text{Asia} & : \quad \sqrt[4]{x^7} = |x^2| \\
\text{Melissa} & : \quad \sqrt[7]{x^4} = x^2
\end{align*}
\]

Who’s correct? Explain your reasoning.
Let’s review the properties of powers.

1. Write an explanation for each property to complete the table.

<table>
<thead>
<tr>
<th>Property of Powers</th>
<th>Rule</th>
<th>Written Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product of Powers</td>
<td>$a^m \cdot a^n = a^{m+n}$</td>
<td></td>
</tr>
<tr>
<td>Quotient of Powers</td>
<td>$\frac{a^m}{a^n} = a^{m-n}$</td>
<td></td>
</tr>
<tr>
<td>Power to a Power</td>
<td>$(a^m)^n = a^{mn}$</td>
<td></td>
</tr>
<tr>
<td>Product to a Power</td>
<td>$(a^m \cdot b^n)^p = a^{mp} \cdot b^{np}$</td>
<td></td>
</tr>
<tr>
<td>Quotient to a Power</td>
<td>$\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$</td>
<td></td>
</tr>
<tr>
<td>Zero Power</td>
<td>$a^0 = 1$, if $a \neq 0$</td>
<td></td>
</tr>
<tr>
<td>Negative Exponent In Numerator</td>
<td>$a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>Negative Exponent In Denominator</td>
<td>$\frac{1}{a^{-m}} = a^m$, if $a \neq 0$ and $m &gt; 0$</td>
<td></td>
</tr>
</tbody>
</table>
2. Cut out the items and tape each item into the appropriate group on the next page.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^0$</td>
<td>$\frac{a^6}{a^6}$</td>
</tr>
<tr>
<td>$(-a)^4$</td>
<td>$\frac{a^3}{a^3}$</td>
</tr>
<tr>
<td>$a^3 \cdot a$</td>
<td>$\frac{a^9}{a^2}$</td>
</tr>
<tr>
<td>$a^0 \cdot a^4$</td>
<td>$\left(\frac{1}{a^2}\right)^2$</td>
</tr>
<tr>
<td>$a \cdot a^{-5}$</td>
<td>$(a^2)^8$</td>
</tr>
<tr>
<td>$a^{-4} \cdot a^0$</td>
<td>$(a^{-12})^\frac{1}{3}$</td>
</tr>
<tr>
<td>$a^4 \cdot a^{-4}$</td>
<td>$\left</td>
</tr>
<tr>
<td>$(a^2)^2$</td>
<td>$\left(\frac{b}{a^2}\right)^2$</td>
</tr>
<tr>
<td>$(ab^2)^2$</td>
<td>$\frac{1}{a^4}$</td>
</tr>
<tr>
<td>$(a^3b^3)^\frac{1}{2}$</td>
<td>$\frac{1}{a^{-4}}$</td>
</tr>
</tbody>
</table>
1

\(a^4\)

\(a^{-4}\)

\(a^2b^4\)

\(a^4b^2\)
You can rewrite a radical as a power with a rational exponent, and rewrite a power with a rational exponent as a radical.

1. Solve the equation $\sqrt{x} = x^a$ for $a$, given $x \geq 0$, to determine the exponential form of $\sqrt{x}$.

   - Square each side of the equation.
   - Because the bases are the same, you can set the exponents equal to each other.
   - Divide by 2 to solve for $a$.

2. Why was the restriction “given $x \geq 0$” stated at the beginning of the worked example?

3. How do you know when the initial $x$-value can be any real number or when the initial $x$-value should be restricted to a subset of the real numbers?
3. Determine the power that is equal to the radical.
   a. Write and solve an equation to determine the power that is equal to the cube root of \( x \).

   b. Write and solve an equation to determine the power that is equal to the cube root of \( x \) squared.

4. Complete the cells in each row. In the last column, write “\( x \geq 0 \)” or “all real numbers” to describe the restrictions that result in equal terms for each row.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Radical to a Power Form</th>
<th>Exponential Form</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt[n]{x^2} )</td>
<td>( (\sqrt[n]{x})^2 )</td>
<td>( x^{\frac{2}{n}} )</td>
<td>( x^{\frac{2}{n}} )</td>
</tr>
<tr>
<td>( \sqrt[x]{x^5} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You can rewrite a radical expression \( \sqrt[n]{x^2} \) as an exponential expression \( x^{\frac{2}{n}} \):

- For all real values of \( x \) if the index \( n \) is odd.
- For all real values of \( x \) greater than or equal to 0 if the index \( n \) is even.
PROBLEM 4 Extracting Roots and Rewriting Radicals

You can extract roots to rewrite radicals, using radicals or powers.

Extract the roots and rewrite $\sqrt[3]{8x^6}$ using radicals and using powers.

<table>
<thead>
<tr>
<th>Using Radicals</th>
<th>Using Powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt[3]{8x^6} = \sqrt[3]{2^3 \cdot x^6}$</td>
<td>$\sqrt[3]{8x^6} = (8x^6)^{\frac{1}{3}}$</td>
</tr>
<tr>
<td>$= 2 \cdot \sqrt[3]{x^6}$</td>
<td>$= (2^3 \cdot x^6)^{\frac{1}{3}}$</td>
</tr>
<tr>
<td>$= 2x^2$</td>
<td>$= 2^1 \cdot x^2$</td>
</tr>
</tbody>
</table>

1. Which method do you prefer?

My motto is, when in doubt rewrite radicals using radical form!
2. Devon and Embry shared their work for extracting roots from $\sqrt[4]{f^2 g^4}$.

- **Embry**
  
  \[
  \sqrt[4]{f^2 g^4} = \sqrt[4]{f^2} \cdot \sqrt[4]{g^4} = (f^2)^{\frac{1}{4}} \cdot (g^4)^{\frac{1}{4}} = f^{\frac{2}{4}} |g| = f^{\frac{1}{2}} |g|
  \]

- **Devon**
  
  \[
  \sqrt[4]{f^2 g^4} = (f^2 g^4)^{\frac{1}{4}} = f^{\frac{2}{4}} g^{\frac{4}{4}} = f^{\frac{1}{2}} g^{1}
  \]

  a. Explain why it is not necessary to use the absolute value symbol around $f^2$.

  b. Explain why it is necessary to use the absolute value symbol around $g$.

In Question 2, Embry extracted the root from $\sqrt[4]{f^2 g^4}$ using radical form because the root of a product is equal to the product of its roots, $\sqrt[n]{a^m b^n} = \sqrt[n]{a^m} \cdot \sqrt[n]{b^n}$.

That concept applies to quotients also. The root of a quotient is equal to the quotient of its roots, $\sqrt[n]{\frac{a^m}{b^n}} = \frac{\sqrt[n]{a^m}}{\sqrt[n]{b^n}}$.
For some radicals, you may not be able to extract the entire radicand.

3. Angelo, Bernadette, and Cris extracted the roots from $\sqrt[3]{16x^8}$.

**Angelo**

\[
\sqrt[3]{16x^8} = (16x^8)^{\frac{1}{3}} \\
= (2^4x^8)^{\frac{1}{3}} \\
= (2^{\frac{4}{3}} \cdot x^8)^{\frac{1}{3}} \\
= 2^{\frac{4}{3} \cdot \frac{1}{3}} \cdot (x^8)^{\frac{1}{3}} \\
= 2^{\frac{4}{9}} \cdot x^{8 \cdot \frac{1}{3}} \\
= 2x^2 \sqrt[3]{2x^2}
\]

**Bernadette**

\[
\sqrt[3]{16x^8} = 2^\frac{4}{3} \cdot x^8 \\
= 2^\frac{1}{3} \cdot (2^3 \cdot x^8 \cdot x^\frac{1}{3}) \\
= 2 \cdot \sqrt[3]{2^3 \cdot x^8 \cdot x^\frac{1}{3}} \\
= 2x^2 \sqrt[3]{2x^\frac{1}{3}}
\]

**Cris**

\[
\sqrt[3]{16x^8} = \sqrt[3]{2^4 \cdot x^8} \\
= \sqrt[3]{2^3 \cdot 2^1 \cdot x^8} \\
= \sqrt[3]{2^3} \cdot 2 \cdot \sqrt[3]{x^8} \\
= 2x^2 \sqrt[3]{2x^8}
\]

a. In the last line of work, why was $2x^2$ not extracted from the radical?

b. Compare and contrast the methods.
4. Betty, Wilma, and Rose each extracted roots and rewrote the radical $\sqrt{x^2 y^2}$.

**Betty**

\[
\sqrt{x^2 y^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|
\]

**Wilma**

\[
\sqrt{x^2 y^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|
\]

**Rose**

\[
\sqrt{x^2 y^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|
\]

Who's correct? Explain your reasoning.
5. Rewrite each radical by extracting all possible roots, and write the final answer in radical form.

a. \( \sqrt{16x^6} \)

b. \(-\sqrt[3]{8v^3}\)

c. \(\sqrt{d^{10}}\)

d. \(\sqrt[6]{h^{12}}\)

e. \(\sqrt{25a^2b^5c^{10}}\)

f. \(\sqrt{81x^5y^{12}}\)

g. \(\sqrt{(x + 3)^9}\)

h. \(\sqrt{(x + 3)^2}\)

Be prepared to share your solutions and methods.
The word radical can describe something that is cool, something that is extreme or very different from the usual, something related to the root or origin in a non-mathematical context, and of course a mathematical function.

The origin or the word radical is related to the Latin word *radix*, meaning “root.”
1. Arianna and Heidi multiplied $\sqrt{18a^2} \cdot 4\sqrt{3a^2}$ and extracted all roots.

**Arianna**

$$\sqrt{18a^2} \cdot 4\sqrt{3a^2} = 4\sqrt{54a^5}$$

$$= 4\sqrt{9 \cdot 6 \cdot a^5}$$

$$= 4 \cdot \sqrt{9} \cdot \sqrt{6} \cdot \sqrt{a^5}$$

$$= 4 \cdot 3 \cdot \sqrt{6} \cdot a^2$$

$$= 12a^2 \sqrt{6}$$

**Heidi**

$$\sqrt{18a^2} \cdot 4\sqrt{3a^2} = \sqrt{9 \cdot 2 \cdot a^2} \cdot \sqrt{3 \cdot a^2}$$

$$= \sqrt{9} \cdot \sqrt{2} \cdot \sqrt{a^2} \cdot \sqrt{3} \cdot \sqrt{a^2}$$

$$= 3 \cdot \sqrt{2} \cdot |a| \cdot 4 \cdot \sqrt{3} \cdot |a|$$

$$= 12a^2 \sqrt{6}$$

Compare Arianna’s and Heidi’s solution methods. Explain the difference in their solution methods.

In a quotient, you can extract roots using different methods.

<table>
<thead>
<tr>
<th>Exponential Form</th>
<th>Radical Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{18a^2}$</td>
<td>$\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$</td>
</tr>
<tr>
<td>$\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$</td>
<td>$\frac{\sqrt{6} \cdot 3a}{4\sqrt{2}a^2}$</td>
</tr>
<tr>
<td>$= \frac{1}{4} \cdot</td>
<td>18a^2</td>
</tr>
<tr>
<td>$= \frac{1}{4} \cdot 6\sqrt{a}$</td>
<td>$= \frac{\sqrt{6}}{4}$</td>
</tr>
<tr>
<td>$= \frac{1}{4} \cdot \sqrt{a}$</td>
<td>$= \sqrt{6}$</td>
</tr>
</tbody>
</table>

I wonder if it would be better to extract roots from each radical first, then divide out common factors?
2. Which method do you think is more efficient?

3. Jackie shared his solution for extracting roots and rewriting the quotient \( \frac{\sqrt{25bc}}{\sqrt{b^2c^2}} \), given \( b > 0 \) and \( c > 0 \).

\[
\begin{align*}
\sqrt{25bc} & = \frac{\sqrt{25} \sqrt{bc}}{\sqrt{b^2c^2}} \\
& = \frac{5}{\sqrt{bc}}
\end{align*}
\]

a. Why are the restrictions \( b > 0 \) and \( c > 0 \), instead of \( b \geq 0 \) and \( c \geq 0 \)?

b. Explain why Jackie's work is incorrect.
c. Robert and Maxine also shared their solutions for extracting roots and rewriting the quotient $\frac{\sqrt{25bc}}{\sqrt{b^2c^2}}$, given $b > 0$ and $c > 0$.

**Robert**

\[
\frac{\sqrt{25bc}}{\sqrt{b^2c^2}} = \frac{\sqrt{25} \cdot \sqrt{bc}}{\sqrt{b^2} \cdot \sqrt{c^2}} = \frac{5\sqrt{bc}}{bc} = \frac{5}{\sqrt{bc}}
\]

**Maxine**

\[
\frac{\sqrt{25bc}}{\sqrt{b^2c^2}} = \frac{\sqrt{25} \cdot \sqrt{bc}}{\sqrt{b^2} \cdot \sqrt{c^2}} = \frac{5\sqrt{bc}}{bc} = \frac{5}{\sqrt{bc}}
\]

Who's correct? Explain your reasoning.
4. Perform each operation and extract all roots. Write your final answer in radical form.
   a. \(2\sqrt{x} \cdot \sqrt{x} \cdot 5\sqrt{x}\), given \(x \geq 0\)
   
   b. \(2\sqrt{k}(\sqrt{k})\)
   
   c. \(7\sqrt{h}(3\sqrt{h} + 4\sqrt{h})\), given \(h \geq 0\)
   
   d. \(\sqrt{a} \cdot \sqrt{a}\), given \(a \geq 0\)
   
   e. \((n\sqrt{4n})(\sqrt{2n^2})\)
   
   f. \(\sqrt[4]{\frac{4x^4}{x^2}}\), given \(x \neq 0\)
5. Perform each operation, extract all roots, and write your final answer in radical form, without radicals in the denominator.

a. \( \frac{2\sqrt{a}}{5\sqrt{a}} \), given \( a > 0 \)

b. \( \frac{2\sqrt{2a^5}}{5\sqrt{16a^2}} \), given \( a > 0 \)

c. \( \frac{-\sqrt{4j^2k^2}}{\sqrt{75jk^2}} \), given \( j > 0 \) and \( k > 0 \)

Multiplying by a form of one helps to eliminate the radical from the denominator.
To add and subtract terms, it is important to identify like terms.

1. Use the symbols to identify six groups of like terms. The first group has been started for you.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4x</td>
<td>$\sqrt{x}$</td>
<td>$\frac{1}{3}x^\frac{1}{5}$</td>
<td>$2x^3$</td>
<td>$0.2x$</td>
</tr>
<tr>
<td>$-8\sqrt{x}$</td>
<td>$10\sqrt[3]{x^2}$</td>
<td>$x^3$</td>
<td>$\frac{3}{\sqrt{x}}$</td>
<td>$\frac{1}{x^\frac{1}{5}}$</td>
</tr>
<tr>
<td>$-7\sqrt[3]{x^2}$</td>
<td>$-6x^\frac{1}{5}$</td>
<td>$-5x^3$</td>
<td>$-3x$</td>
<td>$-\sqrt{x}$</td>
</tr>
<tr>
<td>$-\sqrt[3]{x}$</td>
<td>$\frac{1}{3}\sqrt{x}$</td>
<td>$10\sqrt{x}$</td>
<td>$\frac{3}{\sqrt{x^2}}$</td>
<td>x</td>
</tr>
</tbody>
</table>
In some cases, you can rewrite the sum or difference of two terms as one term.

2. Explain why Grace and Diane were able to rewrite their original expression as one term.

Grace
\[2\sqrt{x} + 6\sqrt{x} = 8\sqrt{x}\]

Diane
\[16\sqrt{x^2} - 10\sqrt{x} = 6\sqrt{x^2}\]

3. Explain why Ron's answer is incorrect.

Ron
\[0.1\sqrt{x} + 3.6\sqrt{x} = 3.7\sqrt{x}\]

Sheila
\[\sqrt{x} + \sqrt{y} = 2\sqrt{xy}\]

4. Explain why Sheila's answer is incorrect.
When adding or subtracting radicals, you can combine like terms and write the result using fewer terms.

For example, the two terms, $3\sqrt{x}$ and $\sqrt{x}$ are like terms because their variable portions, $\sqrt{x}$, are the same. The coefficients do not have to be the same.

On the other hand, the terms $-8\sqrt{x^2}$ and $7\sqrt{x}$ are not like terms because their variable portions, $\sqrt{x^2}$ and $\sqrt{x}$, are different. The indices are the same but the radicands are different. In exponential form, $x^{\frac{1}{2}}$ and $x^{\frac{1}{2}}$, notice that the bases are the same, the denominators in the exponent are the same, but the numerators in the exponents are different.

To determine the sum or difference of like radicals, add or subtract the coefficients.

$$3\sqrt{x} + \sqrt{x} = 4\sqrt{x}, \text{ given } x \geq 0$$

You can also write an equivalent expression using powers.

$$3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{1}{2}}, \text{ given } x \geq 0$$

5. Larry and D.J. discussed whether or not $4\sqrt{x}$ and $-5x^{\frac{1}{2}}$ are like terms, given $x \geq 0$.

Larry
They are not like terms because their variable parts are different.

D.J.
They are like terms. Their variable parts look different, but they are actually the same.

Who’s correct? Explain your reasoning.
6. Combine like terms, if possible, and write your final answer in radical form.

   a. \( \sqrt{y} - \sqrt{y}, \) given \( y \geq 0 \)

   b. \( 9\sqrt{a} + 5\sqrt{b}, \) given \( a \geq 0, b \geq 0 \)

   c. \( 2\sqrt{x} + \sqrt{x} + 5\sqrt{x}, \) given \( x \geq 0 \)

   d. \( 7\sqrt{h} - 4.1\sqrt{h} + 2.4\sqrt{h}, \) given \( h \geq 0 \)

   e. \( 3\sqrt{t}(\sqrt{t} - 8\sqrt{t}) + 4t, \) given \( t \geq 0 \)

   f. \( 5\sqrt{g} + 2\sqrt{g}, \) given \( g \geq 0 \)
Complete the graphic organizer. Write two radicals whose sum, difference, product, and quotient are each equivalent to $6 \sqrt{x}$.

Be prepared to share your solutions and methods.
So, you have been wondering whether there is a system to measure wind speed and describe conditions at sea and on land, right? The answer is the Beaufort scale. It was developed in the early 1800s and is still in use today.

### Beaufort Scale

<table>
<thead>
<tr>
<th>Beaufort Number</th>
<th>Description</th>
<th>Wind Speed (miles per hour)</th>
<th>Wave Height (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>calm</td>
<td>&lt; 1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>light air</td>
<td>1–3</td>
<td>0–1</td>
</tr>
<tr>
<td>2</td>
<td>light breeze</td>
<td>4–7</td>
<td>1–2</td>
</tr>
<tr>
<td>3</td>
<td>gentle breeze</td>
<td>8–13</td>
<td>2–3.5</td>
</tr>
<tr>
<td>4</td>
<td>moderate breeze</td>
<td>14–17</td>
<td>3.5–6</td>
</tr>
<tr>
<td>5</td>
<td>fresh breeze</td>
<td>18–24</td>
<td>6–9</td>
</tr>
<tr>
<td>6</td>
<td>strong breeze</td>
<td>25–30</td>
<td>9–13</td>
</tr>
<tr>
<td>7</td>
<td>near gale</td>
<td>31–38</td>
<td>13–19</td>
</tr>
<tr>
<td>8</td>
<td>gale</td>
<td>39–46</td>
<td>18–25</td>
</tr>
<tr>
<td>9</td>
<td>strong gale</td>
<td>47–54</td>
<td>23–32</td>
</tr>
<tr>
<td>10</td>
<td>storm</td>
<td>55–63</td>
<td>29–41</td>
</tr>
<tr>
<td>11</td>
<td>violent storm</td>
<td>64–73</td>
<td>37–52</td>
</tr>
<tr>
<td>12</td>
<td>hurricane</td>
<td>74</td>
<td>46</td>
</tr>
</tbody>
</table>
PROBLEM 1 Analyzing Solution Paths for Radical Equations

Strategies for solving equations such as maintaining balance and isolating the term containing the unknown are applicable when solving radical equations.

Let’s compare the algebraic solution of a two-step quadratic equation to a two-step radical equation.

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<th>Solution Steps for a Radical Equation</th>
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<td>$2x^2 - 5 = 13$</td>
<td>$2\sqrt{x} - 5 = 13$</td>
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<td>Check $x = 3$:</td>
<td>Check $x = 81$:</td>
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<td>$2x^2 = 18$</td>
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<td>$x^2 = 9$</td>
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<td>$x = \pm 3$</td>
<td>$x = 81$</td>
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<tr>
<td>$2(-3)^2 - 5 \leq 13$</td>
<td>$13 = 13\checkmark$</td>
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1. Analyze the examples.
   a. Describe the similarities in the first two steps of each solution.

   b. Describe the differences in the remaining steps of each solution.

2. How would the strategy shown in the worked example change for cube and cube root equations? Provide an example to explain your reasoning.
3. Franco, Theresa, Dawnelle, and Marteiz shared their work for solving $3\sqrt{x} + 7 = 25$, given $x \geq 0$.

**Franco**

$$3\sqrt{x} + 7 = 25$$

$$(3\sqrt{x} + 7)^2 = (25)^2$$

$$9x + 42\sqrt{x} + 49 = 625$$

$$9x + 42\sqrt{x} = 576$$

$$3(3x + 14\sqrt{x} - 192) = 0$$

$$3(3\sqrt{x} + 32)(\sqrt{x} - 6) = 0$$

$$3\sqrt{x} + 32 = 0 \text{ or } \sqrt{x} - 6 = 0$$

$$3\sqrt{x} = -32 \quad \sqrt{x} = 6$$

$$\sqrt{x} = -\frac{32}{3} \quad (\sqrt{x})^2 = (6)^2$$

$$\sqrt{x} = \frac{32}{3} \quad x = 36$$

$$x = \frac{1024}{9}$$

**Theresa**

$$3\sqrt{x} + 7 = 25$$

$$(3\sqrt{x} + 7)^2 = (25)^2$$

$$9x + 42\sqrt{x} + 49 = 625$$

$$9x + 42\sqrt{x} = 576$$

$$3(3x + 14\sqrt{x} - 192) = 0$$

$$3(3\sqrt{x} + 32)(\sqrt{x} - 6) = 0$$

$$3\sqrt{x} + 32 = 0 \text{ or } \sqrt{x} - 6 = 0$$

$$3\sqrt{x} = -32 \quad \sqrt{x} = 6$$

$$\sqrt{x} = -\frac{32}{3} \quad (\sqrt{x})^2 = (6)^2$$

$$\sqrt{x} = \frac{32}{3} \quad x = 36$$

$$x = \frac{1024}{9}$$

Check:

$$3\sqrt{\left(\frac{1024}{9}\right)} + 7 \neq 25$$

$$3\sqrt{36} + 7 \neq 25$$

$$3\left(\frac{32}{3}\right) + 7 \neq 25$$

$$3(6) + 7 \neq 25$$

There is one solution, $x = 36$.

**Dawnelle**

$$3\sqrt{x} + 7 = 25$$

$$(3\sqrt{x} + 7)^2 = (25)^2$$

$$3x + 7 = 625$$

$$3x = 618$$

$$x = 206$$

**Marteiz**

$$3\sqrt{x} + 7 = 25$$

Check:

$$3\sqrt{x} = 18$$

$$\sqrt{x} = 6$$

$$\sqrt{x} = \frac{32}{3} \quad (\sqrt{x})^2 = (6)^2$$

$$x = 36$$

$$x = 25$$

$$25 = 25 \checkmark$$
a. Theresa and Marteiz each solved the equation correctly. Explain the difference between their solution methods.

b. Explain the error in Franco's work.

c. Explain the error in Dawnelle's work.

4. John says that to solve an equation with rational exponents, you can rewrite terms in radical form and then solve.

\[ 2x^{\frac{1}{3}} - 5 = 3 \quad \rightarrow \quad 2 \sqrt[3]{x} - 5 = 3 \]

Is John correct? What is the solution to the equation? Explain your reasoning.
5. Solve and check each equation.
   a. \( \sqrt{2x} = 3 \)
   
   b. \( \sqrt{2x} - 3 = 2 \)
   
   c. \( 4\sqrt{x} - 6 = 8 \)
   
   d. \( \sqrt{2x} + 1 = 5 \)
   
   e. \( 2\sqrt{x} + 16 = 0 \)
   
   f. \( (3x - 1) \frac{1}{2} + 9 = 8 \)
g. \( x - x^\frac{1}{2} = 2 \)

h. \( x - 1 = \sqrt{x + 1} \)
1. The Beaufort scale is a system that measures wind speed and describes conditions at sea and on land. The scale’s range is from 0 to 12. A zero on the Beaufort scale means that the wind speed is less than 1 mile per hour and the conditions at sea and on land are calm. A twelve on the Beaufort scale represents hurricane conditions with wind speeds greater than 74 miles per hour, resulting in greater than 50-foot waves at sea and severe damage to structures and landscape.

Consider the equation \( V = 1.837B^2 \) that models the relationship between wind speed in miles per hour \( V \) and the Beaufort numbers \( B \). Determine the Beaufort number for a wind speed of 20 miles per hour.

2. In medicine, Body Surface Area \( BSA \) is used to help determine proper dosage for medications. The equation \( BSA = \frac{\sqrt{W \cdot H}}{60} \) models the relationship between \( BSA \) in square meters, the patient’s weight \( W \) in kilograms, and the patient’s height \( H \) in centimeters. Determine the height of a patient who weighs 90 kilograms and has a \( BSA \) of 2.1.
3. Big Ben is the nickname of a well-known clock tower in London, England, that stands 316 feet tall. The clock is driven by a 660-pound pendulum in the tower that continually swings back and forth. The relationship between the length of pendulum $L$ in feet and the time it takes for a pendulum to swing back and forth one time, or its period $T$, is modeled by the equation $T = 2\pi\sqrt{\frac{L}{32}}$. If the pendulum's period is 4 seconds, determine the pendulum's length.
4. A pilot is flying a plane high above the earth. She has clear vision to the horizon ahead.

a. Use the diagram to derive an equation to show the relationship between the three sides of the triangle. Then, solve the equation for the plane’s altitude, \( p \).

Note: The variable \( r \) represents the Earth’s radius (miles), \( p \) represents the plane’s height above the earth, or altitude (miles), and \( h \) represents the distance from the pilot to the horizon (miles).
b. Use your equation from part (a) to calculate the plane’s altitude, if the distance from the pilot to the horizon is 225 miles. The earth’s radius is 3959 miles.

Be prepared to share your solutions and methods.
KEY TERMS

- inverse of a function (10.1)
- invertible function (10.1)
- Horizontal Line Test (10.1)
- square root function (10.2)
- cube root function (10.2)
- radical function (10.2)
- composition of functions (10.2)

10.1 Graphing Inverses of Power Functions

A function $f$ is the set of all ordered pairs $(x, y)$ or $(x, f(x))$, where for every value of $x$, there is one and only one value of $y$, or $f(x)$. The inverse of a function is the set of all ordered pairs $(y, x)$, or $(f(x), x)$. To graph the inverse of a function, simply reflect the function over the line $y = x$.

Example

Graph $f(x) = x^3$, and then graph its inverse.

![Graph of function and its inverse](image-url)
## 10.1 Determining Whether or Not Functions are Invertible

To determine whether or not a function is invertible, graph the function and apply the Horizontal Line Test. If the graph of the function passes the Horizontal Line Test, then it is invertible.

**Example**

Determine whether or not \( f(x) = \frac{x^4}{56} \) is invertible.

The function \( f(x) = \frac{x^4}{56} \) is not invertible, because it fails the Horizontal Line Test. That is, a horizontal line can pass through more than one point on the graph at the same time.

## 10.2 Determining the Equation for the Inverse of a Power Function

To determine the equation for the inverse of a power function, transpose the \( x \) and the \( y \) in the equation and then solve for \( y \).

**Example**

Determine the equation for the inverse of the function \( y = \frac{2}{3} x^3 \).

\[
\begin{align*}
  y &= \frac{2}{3} x^3 \\
  x &= \frac{2}{3} y^3 \\
  \frac{3}{2} x &= y^3 \\
  \frac{\sqrt[3]{3}}{2} x &= y \\
  \frac{\sqrt[3]{48x}}{2} &= y
\end{align*}
\]

Therefore, the equation for the inverse of the function \( y = \frac{2}{3} x^3 \) is \( y = \frac{\sqrt[3]{48x}}{2} \).
10.2 Describing the Characteristics of Square Root and Cube Root Functions

The characteristics of square root and cube root functions include the domain, range, and \(x\)- and \(y\)-intercepts.

Example

Describe the characteristics of the function \(f(x) = \sqrt{5 - x}\)

Domain: \((-\infty, 5]\)

Range: \([0, \infty)\)

\(x\)-intercept: \((5, 0)\)

\(y\)-intercept: \((0, \sqrt{5})\)

10.3 Describing Transformations of Radical Functions

Transformations performed on a function \(f(x)\) to form a new function \(g(x)\) can be described by the transformational function:

\[g(x) = Af(B(x - C)) + D\]

Translating a Radical Function Horizontally: If a number, \(C\), is added under the radical, the graph of the function is shifted \(C\) units to the left. If a number, \(C\), is subtracted under the radical, the graph of the function is shifted \(C\) units to the right.

Translating a Radical Function Vertically: If a number, \(D\), is added outside the radical, the graph of the function is shifted \(D\) units up. If a number, \(D\), is subtracted outside the radical, the graph of the function is shifted \(D\) units down.

Vertically Stretching and Compressing a Radical Function: Multiplying the function by a number, \(A\), that is greater than one vertically stretches the function. Multiplying the function by a number, \(A\), that is greater than zero but less than one vertically compresses the function.

Reflecting a Radical Function: Multiplying the function by a negative one reflects the graph across the \(x\)-axis. Multiplying by a negative one under the radical reflects the graph across the \(y\)-axis.

Example

Describe how the graph of the function \(f(x) = \sqrt[3]{x}\) would be transformed to produce the graph of the function \(g(x) = 2f(x - 4) + 1\).

The graph of \(f(x)\) would be vertically stretched by a factor of two, translated 4 units to the right and up 1 unit.
10.3 Graphing Transformations of Radical Functions

Transformations that take place inside the radical shift the function left or right. Transformations that take place outside the radical shift the function up or down.

Example

The graph of \( f(x) = \sqrt{x} \) is shown. Graph the transformation of \( f(x) \) as represented by the equation \( g(x) = f(x + 5) + 3 \). Then, list the domain for each function.

![Graph of functions](image)

Domain of \( f(x) \): \([0, \infty)\)
Domain of \( g(x) \): \([-5, \infty)\)

10.4 Rewriting Radical Expressions

To rewrite a radical expression, extract the roots by using the rational exponents and the properties of powers. To extract a variable from a radical, the expression \( \sqrt[n]{x^n} \) can be written as \(|x|\) when \( n \) is even, and \( x \) when \( n \) is odd.

Example

Rewrite the expression \( \sqrt[4]{625x^8y^5z} \).

\[
\sqrt[4]{625x^8y^5z} = \sqrt[4]{625} \cdot x^2 \cdot y \cdot y \cdot z
\]

\[
= 5x^2 |y| \sqrt[4]{5y} \cdot \sqrt[4]{y} \cdot \sqrt[4]{z}
\]

\[
= 5x^2 |y| \sqrt[4]{5yz}
\]
Operating with Radicals

To operate on radicals, follow the order of operations and properties of powers. Remember to extract all roots.

Example

Perform the indicated operations and extract all roots for \( x \geq 0 \) and \( y \geq 0 \). Write your final answer in radical form.

\[
(-3x\sqrt{x^2y^6})(2\sqrt{x^3}) - 5\sqrt[4]{81x^{20}y^{12}} = \left(-3x(x^3y^6)^{\frac{1}{2}}\right)\left(2(x^3)^{\frac{3}{2}}\right) - 5(81x^{20}y^{12})^{\frac{1}{4}}
\]

\[
= (-3 \cdot x \cdot x^\frac{5}{2} \cdot y^3)(2 \cdot x^\frac{3}{2}) - (5 \cdot 81^\frac{1}{4} \cdot x^{20} \cdot y^{12})^\frac{1}{4}
\]

\[
= (-3 \cdot x^7 \cdot y^3)(2 \cdot x^\frac{3}{2}) - (5 \cdot 3 \cdot x^5 \cdot y^3)
\]

\[
= -6 \cdot x^5 \cdot y^3 - 15 \cdot x^5 \cdot y^3
\]

\[
= -21x^5y^3
\]

Solving Radical Equations

To solve a radical equation, isolate the radical term if possible. Then, raise the entire equation to the power that will eliminate the radical. Finally, follow the steps necessary to solve the equation. Check for extraneous solutions.

Example

\[
\sqrt{x} + 2 + 10 = x
\]

Check:

\[
\sqrt{14} + 2 + 10 \neq 14
\]

\[
\sqrt{9} + 10 \neq 14
\]

There is one solution, \( x = 14 \).
Problem Solving with Radical Equations

To solve a problem with radical equations, identify what the problem is asking. Then, determine how to use the given equation to solve the problem. Finally, follow the process for solving radical equations.

Example

The distance between any two points on a coordinate plane can be calculated by using the equation \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \), where \((x_1, y_1)\) represents the coordinates of one point and \((x_2, y_2)\) represents the coordinates of the other point. Determine the point(s) on the line \( y = 1 \) that is (are) exactly 5 units from the point \((1, -2)\). Use the point \((x, 1)\) to represent a point on the line \( y = 1 \).

\[
5 = \sqrt{(1 - x)^2 + (-2 - 1)^2}
\]
\[
5 = \sqrt{1 - 2x + x^2 + 9}
\]
\[
25 = 1 - 2x + x^2 + 9
\]
\[
0 = x^2 - 2x - 15
\]
\[
0 = (x + 3)(x - 5)
\]
\[
x = -3 \text{ or } x = 5
\]

The points \((-3, 1)\) and \((5, 1)\) are on the line \( y = 1 \) and are exactly 5 units from the point \((1, -2)\).